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Entropy on Charge Density: Making the Quantum Mechanical Connection

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Abstract

Entropy maximization has proven effective in treating certain aspects of the phase problem of X-ray diffraction. Much of its development has been expressed in probabilistic language, although image enhancement has been somewhat more physical or geometric in description. Here phasing and entropy maximization are embedded in the quantum mechanical problem of reconstructing an electronic one-matrix under experimental constraints. Entropy on an N-representable one-particle density matrix is well defined. The entropy is the expected form, and it is a simple function of the one-matrix eigenvalues which all must be non-negative. Certain other properties are pertinent to phasing which is implicit in one-matrix reconstruction governed by entropy maximization. Throughout this work reference is made to informational entropy, not the entropy of thermodynamics.

1. Introduction

Electron density is the outcome of solving an X-ray crystallographic phase problem. While structure-factor phases are commonly viewed as the desired result of phasing calculations, they have no substantive use other than in generation of a charge density through Fourier synthesis. This is especially evident in macromolecular crystallography where the penultimate step in solution of a phase problem is expert judgment that the derived density function is indeed that of, say, a protein, and the implicit final judgment is that an acceptable structure model has been developed.

Electronic charge density is also a property calculated in quantum mechanical studies of chemical systems, and it plays a central role in their discussion. This centrality of charge density was given formal standing by the Hohenberg-Kohn theorem (HKT hereafter) (1964). An informal summary of HKT is that charge density alone suffices to fully delimit the ground state of a non-degenerate electronic system. Clearly, such idealization is completely out of the range of experimental protein crystallography. Nevertheless, expressing the phase problem in quantum mechanically informed constructions may uncover or at least open the way to additional power for its solution.

The phase problem is expressed here, not in probabilistic language, but as a physics problem of diffraction and quantum theory, which is to be solved by resolution of available data and constraints arising from the problem's physical nature. Density matrices will be the principal tool for this, and their computation and manipulation will be governed by constrained entropy maximization. This entropy, in spite of the physics context, is not the entropy of thermodynamics, but informational entropy (Skilling, 1988). Physical character is brought to the problem not by entropy, which here serves only as an objective function for optimization, but by the density-matrix structure that quantum mechanics requires.

2. Density matrices

The connection between quantum mechanics and crystallography centers on charge density. Its representation in a density-matrix formalism is particularly useful because of the consequent analytic relationships between direct and reciprocal space. An excellent discussion of these relationships and the construction of density matrices has been given recently by Schmider, Smith & Weyrich (1992).

Given a wavefunction Ψ for a non-degenerate electronic ground state of N particles, the density matrix closest to Ψ is

$$\Gamma(\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}') = \Psi(\mathbf{x},\mathbf{y})\Psi^*(\mathbf{x}',\mathbf{y}'), \qquad (1)$$

a dyadic product (Davidson, 1976) in which, for convenience, the coordinates are partitioned such that $\mathbf{x}$ collects the four space-spin coordinates of electron one, and $\mathbf{y}$ collects the coordinates of the other $N-1$ electrons. A reduced density matrix is found by contraction of Γ through averaging over the coordinates not of interest. Of particular interest here is the reduced one-matrix for which common notations are $\gamma(1,1')$, where the number 1 stands for the four space-spin coordinates of electron one, or $\rho_1(\mathbf{x},\mathbf{x}')$, and, *inter alia*, $\rho_1(\mathbf{x},\mathbf{x}')$ may be regarded as the kernel of an integral operator ρ_1 (McWeeny & Sutcliffe, 1969). The eight-dimensional one-matrix in terms of Ψ is

$$\rho_1(\mathbf{x},\mathbf{x}') = N\int \Psi(\mathbf{x},\mathbf{y})\Psi^*(\mathbf{x}',\mathbf{y})\,\mathrm{d}\mathbf{y}, \qquad (2)$$

a continuous function of the two four-dimensional vectors $\mathbf{x}$ and $\mathbf{x}'$, and the corresponding spin-free or spin-traced density matrix is $\rho(\mathbf{r},\mathbf{r}')$. The density matrix is hermitian, and it is termed a one-matrix in recognition that the surviving coordinates are for any one electron of the N-electron ensemble.

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One-particle density matrices carry sufficient information for all one-electron properties (McWeeny & Sutcliffe, 1969). Charge density, although clearly dependent on the disposition of all electrons in a multiparticle system, is a (spin-free) one-electron property. It is the probability density of finding, in the presence of the other electrons, any one of the N electrons at $\mathbf{r}$, and it is given by the so-called diagonal of $\rho(\mathbf{r},\mathbf{r}')$, that portion for which $\mathbf{r} = \mathbf{r}'$, and $\rho(\mathbf{r},\mathbf{r})$ is written as $\rho(\mathbf{r})$, which is the ordinary crystallographic electron density at $\mathbf{r}$.

It is useful to expand the one-matrix in an orthogonal basis. In terms of an arbitrary set of basis functions φ_i,

$$\rho(\mathbf{r},\mathbf{r}') = \sum_{ik} P_{ik}\, \varphi_i(\mathbf{r})\, \varphi_k^*(\mathbf{r}'), \tag{3}$$

and the square array $\mathbf{P}$ is its number representation. The (spin-free) one-matrix is most compactly expressed in the eigenvalue equation

$$\rho(\mathbf{r},\mathbf{r}') = \sum_{j=1}^{\infty} n_j\, \chi_j(\mathbf{r})\, \chi_j^*(\mathbf{r}'), \tag{4}$$

in which an eigenvalue n_j is the occupation number of (spin-free) natural orbital χ_j. Natural orbitals are the eigenfunctions of the one-matrix, and the ordinary electron density may be written

$$\rho(\mathbf{r}) = \sum_{j=1}^{\infty} n_j\, |\chi_j(\mathbf{r})|^2. \tag{5}$$

One requirement for a density-matrix representation of a fermion system is that it be N-representable. This expresses the necessity that a density matrix be derivable from an N-particle dyadic product as in (2). For a (fermion) one-matrix to be N-representable it suffices that its eigenvalues sum to N and lie in the range [0, 1] (Coleman, 1963). In the approximation of electrons as independent particles, the eigenvalues are all either 0 or 1 and the one-matrix is idempotent. For a spin-traced one-matrix, it may be necessary to consider the eigenvalue range [0, 2] to allow for double occupancy of orbitals; the eigenvalues must always sum to N.

Useful approximations for many quantum mechanical problems are based on the assumption of independent or non-interacting particles. Systems of independent electrons are described by idempotent density matrices and represent the Hartree-Fock or self-consistent field limit of quantum mechanical calculation. While even this approximation is well out of reach of protein crystallography for the foreseeable future, its feature of matrix idempotence has great practical importance.

Idempotent matrices have eigenvalues which are 0 or 1, and can be characterized by the matrix equation $\mathbf{P} = \mathbf{PP}$. If the product of two integral operators with kernels $A(\mathbf{x},\mathbf{x}')$, $B(\mathbf{x},\mathbf{x}')$, is the integral operator with kernel $C(\mathbf{x},\mathbf{x}')$,

$$C(\mathbf{x},\mathbf{x}') = \int A(\mathbf{x},\mathbf{x}'')B(\mathbf{x}'',\mathbf{x}')\, d\mathbf{x}'', \tag{6}$$

then the associated matrices satisfy $\mathbf{C} = \mathbf{AB}$, and if the one-electron operator ρ_1 has the property $\rho_1 = \rho_1^2$, it is a projection operator and its matrix representation is idempotent. The corresponding eigenvalue equation becomes

$$\rho(\mathbf{r},\mathbf{r}') = \sum_{j=1}^{N} \chi_j(\mathbf{r})\, \chi_j^*(\mathbf{r}'), \tag{7}$$

and although the general expansion of (3) is not simplified, its number representation $\mathbf{P}$ must possess the properties of idempotence. This points directly to the computational practicality of dealing with density matrices, namely that they can be manipulated through their number representations apart from realization in an appropriate orthogonal basis.

3. Entropy on a density matrix

Jaynes (1957) gave entropy on a density matrix in the form

$$S = -\mathrm{tr}[\rho(\mathbf{r},\mathbf{r}') \ln \rho(\mathbf{r},\mathbf{r}')], \tag{8}$$

the negative trace, which may be computed as

$$S = -\sum_{j=1}^{\infty} n_j \ln n_j, \tag{9}$$

where the eigenvalues n_j are those of (4). It is clear that entropy on an N-representable fermion density matrix is well defined and can be calculated.

A density matrix can be constructed from structure factors by rearrangement of

$$\rho(\mathbf{r}) = \frac{1}{V} \sum_i F(\mathbf{h}_i) \exp[-2\pi i \mathbf{h}_i \cdot \mathbf{r}], \tag{10}$$

the Fourier synthesis of charge density. For a crystal unit cell of volume V, define

$$T_{ik} \equiv T_{\mathbf{h}_i - \mathbf{h}_k}; \quad \xi_i(\mathbf{r}) \equiv V^{-1/2} \exp\{-2\pi \mathbf{h}_i \cdot \mathbf{r}\}. \tag{11}$$

Then if by any assignment

$$F_{\mathrm{h}} = \mathcal{F}(\mathbf{h},\mathbf{T}) \equiv \sum_{\mathbf{h}_i - \mathbf{h}_k = \mathbf{h}} T_{ik}, \tag{12}$$

$$\rho(\mathbf{r},\mathbf{r}') = \sum_{ik} T_{ik}\, \xi_i(\mathbf{r})\, \xi_k^*(\mathbf{r}') \tag{13}$$

is a full density matrix carrying the charge density as its diagonal. Because the elements T_{ij} are restricted only to satisfy (12), (13) cannot be expected to give the true density matrix. One additional completely general constraint may be imposed on any real electronic system; its one-matrix must be positive semidefinite, and assignment of values for $\mathbf{T}$ must give a matrix with non-negative eigenvalues. The other general constraint has already been satisfied by (12) by the fact that F_0 is, on an absolute scale, the number of electrons in a unit cell, and the one-matrix eigenvalues must therefore sum to N.

Under these conditions entropy is well defined, both on the density matrix and its diagonal, the ordinary charge density, so long as it also is non-negative. Another well defined and useful functional is

$$Q = \sum_{j=1}^{\infty}(n_j - n_j^2), \tag{14}$$

which has strong parallels with entropy. Some of the parallels are (negative eigenvalues disallowed):

(1) Q and S are both zero for idempotent matrices.

(2) Q and S are both increased by any pairwise change which preserves $\sum_j n_j$ and diminishes $\sum_j n_j^2$. In the case that the Q and S regularization formulas are applied to an ordinary non-negative density function of fixed mean value, the flatter the density, the greater both Q and S (Collins & Prince, 1991).

(3) Q and S both have a maximum for a fixed number of positive eigenvalues when the positive eigenvalues are all equal, and any others are zero.

In the sense of these parallels, Q and S are equivalent measures for flatness of a density map or an eigenvalue spectrum. No significance is ascribed to Q beyond its role as a convenient computational surrogate for S.

It should be noted that $\mathbf{T}$ is not a Karle–Hauptman matrix (Karle & Hauptman, 1950). In such a matrix each different entry with a subscript pair following the pattern $i, k \Rightarrow \mathbf{h}_i - \mathbf{h}_k = \mathbf{h}$ has the same value, namely $F_{\mathbf{h}}$. In $\mathbf{T}$, as given by (12), all different entries associated with $\mathbf{h}$ are required only to sum to $F_{\mathbf{h}}$; they are not required to have the same value and they are expected to be unequal. For its initial construction, given a phase set assignment, one practical choice of first values for the elements T_{ik}, and the one used in test calculations, is $F_{\mathbf{h}}$ divided by the number of its occurrences in $\mathbf{T}$.

4. Entropy maximization

Macromolecular crystallography generally involves unit cells containing at least tens of thousands of electrons. For these cases computationally tractable number representations of density matrices are not likely to have a rank as large as N in $\mathbf{T}$. Of course if the rank of $\mathbf{T}$ is smaller than the number of electrons in a unit cell, the eigenvalues of $\mathbf{T}$, which must sum to N, will violate their N-representability upper bound, and (13) can never be better than an approximation to a true fermion density matrix. Nevertheless, if value assignments in $\mathbf{T}$ are governed by maximization of the entropic regularization formula for S, and are constrained by $|\mathcal{F}(\mathbf{h}, \mathbf{T})| = |F_{\mathbf{h}}|$, the approximate density-matrix diagonal will suffer only the usual imperfections of data limitation for the given phase assignment, and the eigenvalues of $\mathbf{T}$ will be as nearly equal as possible. In the following discussion it is assumed that the rank of $\mathbf{T}$ is significantly less than N.

One way to maximize entropy on $\mathbf{T}$ is to adjust the permissible assignment of values to T_{ik} so that $\mathbf{T} = \kappa\mathbf{TT}$, where κ is a positive constant; in such a case $\mathbf{T}$ will be termed idempotent within a scale factor, or just idempotent if the need for scaling is clear. The eigenvalues for idempotent $\mathbf{T}$ are all zero or κ^{-1}, and the regularizations S and Q are a maximum. In application it is not anticipated that $\mathbf{T}$ can be made idempotent within a scale factor, under the conditions of (12), if its rank is less than N. In any case, when constrained Q is made a maximum by adjustment of T_{ik}, the similarly constrained entropy S is moved toward its maximum in the sense that Q provides a lower bound to S (Harrison, 1987).

To verify that maximizing entropy on $\mathbf{T}$ could be useful in dealing with the phase problem, the lower bound maximization of Q was undertaken with real data for bovine pancreatic phospholipase A$_2$ (Dijkstra, Kalk, Hol & Drenth, 1981). In these preliminary tests the phospholipase was treated as though its space group were $P1$. An initial matrix $\mathbf{T}$ was constructed as described at the end of the previous section, and driven toward idempotence in a minimization of

$$\sum_{i,k} |T_{ik}^{\mathrm{sq}} - T_{ik}|^2 + \sum_{\mathbf{h}} |\mathcal{F}(\mathbf{h}, \mathbf{T}) - F_{\mathbf{h}}|^2 \tag{15}$$

by a truncated-Newton method (Nash, 1985); $\mathbf{T}^{\mathrm{sq}} = \mathbf{TT}$. In the largest trial 24 reflections of lowest $\sin\theta/\lambda$ were used for $F_{\mathbf{h}_i}$, thus the rank of $\mathbf{T}$ was 24, corresponding to its complex entries the number of real variables in the optimization was 576, and the number of phases returned was about 100. Several trials were run for ranks ranging between 15 and 24. On average, about 67% of the returned phases were within 10° of the correct values, about 5% were in error by more than 45°, and the average phase error was about 34°. In each case the gradient vector magnitude was reduced by nine or more orders of magnitude as the minimization progressed. Clearly, these tests provide no positive information about the phasing power of entropy maximization on density matrices. Because they were initiated with the correct phases, and the minimizations required minimal departure from them, the tests provide only the non-negative result that raising the lower bound of entropy on a density matrix in a constrained maximization of Q can be carried out without destroying proposed correct phase information.

As efficient algorithms for these calculations are constructed, the rank of $\mathbf{T}$ can be increased substantially. In the limit of rank $\rightarrow N$, the number of electrons in a unit cell, it is known that a physical density is representable by a system of non-interacting particles (Kohn, 1985), and consequently by an idempotent density matrix. Evidently as the rank of $\mathbf{T}$ increases, the lower bound on S established by maximum Q becomes tighter until at idempotence, as pointed out earlier, Q and S simultaneously achieve their respective maxima. On ordinary electron density, or a density-matrix diagonal, $Q \leq S$ and Q is maximized when $|\rho(\mathbf{r}) - \rho^2(\mathbf{r})|$ is minimized (Harrison, 1987); $Q = S$ when the density function is absolutely flat. On a full density matrix, in contrast, $Q = S$ however highly structured the ordinary density may be, if in an operator equation $\rho = \rho^2$ and the one-matrix is idempotent.

Equation (9), which is the entropy formula in terms of eigenvalues, shows that entropy maximization is most directly achievable through control of the eigenvalues of $\rho(\mathbf{r}, \mathbf{r}')$. Because the one-matrix is hermitian, **T** can be resolved as

$$\mathbf{T} = \mathbf{C}\mathbf{N}\mathbf{C}^{\dagger}, \qquad (16)$$

where **C** is the unitary eigenvector matrix of **T**, and **N** is the diagonal matrix of eigenvalues. Schmider, Smith & Weyrich (1992) propose building **C** from a series of Jacobi rotations and reconstructing **T** in an optimization parameterized by the rotations α_i and the eigenvalues n_j. Of course, in this work the optimization would be entropy maximization, experimental constraints would include $|\mathcal{F}(\mathbf{h}, \mathbf{T})| = |F_{\mathbf{h}}|$, and successful reconstruction of **T** under these conditions would constitute a complete solution to the phase problem.

5. Concluding remarks

In the past decade, entropy maximization has proven to be a powerful tool in dealing with various aspects of the crystallographic phase problem. Although use of informational entropy has been developed from many points of view, its most common expression has been in probabilistic terms. Here a new approach has expressed entropy maximization and phasing in the physical context of molecular quantum chemistry. Indeed, in the spirit of HKT, one might say that phasing by entropy maximization on a one-matrix is equivalent to using structure-factor moduli to establish the full ground state of an isolated system of interacting electrons (Levy & Perdew, 1985). Of course HKT is at present only an existence theorem, and a current area of research is formation of the true off-diagonal portion of a one-matrix when the diagonal portion is known. Entropy maximization on a one-matrix provides a formalism for connecting its diagonal and off-diagonal portions and presents the phase problem as a physical problem of quantum mechanics.

Solution of the phase problem by entropy maximization on a one-matrix ensures quantum mechanical correctness of the relationship between structure moduli and the charge density implied by phase determination. Moreover, the density-matrix formalism allows for straightforward inclusion in the phasing process of other kinds of information, *e.g.* momentum density constraints whether from theoretical considerations or inelastic scattering experiments (Schmider, Smith & Weyrich, 1992). Because density matrices are routinely reducible to eigen representations, demonstrable optimality is expected among related algorithms for dealing with the phase problem.

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